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Spatio-Temporal Gradient Analysis of Differential Microphone Arrays

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ABSTRACT

The literature on gradient and differential microphone arrays makes a distinction between the two types, and nevertheless shows how both types can be used to obtain the same directional responses. A more theoretically sound rationale for using delays in differential microphone arrays has not yet been given. This paper presents a gradient analysis of the sound field viewed as a spatio-temporal phenomenon, and gives a theoretical interpretation of the working principles of gradient and differential microphone arrays. It shows that both types of microphone arrays can be viewed as devices for approximately measuring spatio-temporal derivatives of the sound field. Furthermore, it also motivates the design of high-order differential microphone arrays using the aforementioned spatio-temporal gradient analysis.

1. INTRODUCTION

The literature on gradient microphones dates back to the middle of the last century and the works of Olson (e.g., see [1,2]), who described gradient microphones as arrays of pressure sensing elements whose signals are combined in the similar way gradients are approximated with finite differences. The works of Olson [1, 2] also showed how the combination of responses from gradient microphones of different order can be equivalently obtained by combining signals from multiple pressure microphones with appropriately chosen delays. Later, the microphone arrays of the latter type, which use delay elements, has been termed differential microphone arrays, and have been more extensively analyzed in the works of Elko at the start of the last decade (e.g., see [3, 4]). However, although intuitively present, the relation between these two microphone array types, and the rationale behind the idea of combining the delayed microphone signals in differential microphone arrays, has not been given yet.

This paper presents a slightly different analysis of the sound pressure field, which is viewed as both a spatial and a temporal phenomenon, i.e., as a multivariate function of spatial location and time. This analysis then exposes the operations of taking gradients and directional derivatives of the sound pressure field as combinations of its spatial and temporal derivatives. This new perspective gives a clear interpretation of gradient and differential microphones and microphone arrays: the former as devices used for approximately measuring only spatial derivatives, and the latter as devices used for approximately measuring spatio-temporal derivatives of the sound pressure field. In other words, it shows their equivalence.

This paper is organized as follows. Section 2 gives a theoretical analysis of spatio-temporal derivatives of the sound pressure field when the latter is viewed as both a spatial and a temporal phenomenon, i.e., as a multivariate function of space and time. Section 3 shows a number of practical differential microphone arrays which follow from the theoretical analysis from Section 2. Conclusions are given in Section 4.

2. THEORETICAL ANALYSIS OF SPATIO-TEMPORAL DERIVATIVES OF THE SOUND FIELD

2.1. Sound pressure field's spatial derivatives

The sound pressure at a position defined by vector $\mathbf{r}$ of a plane wave propagating with wave vector $\mathbf{k}$ is given by [5]

$$p(\mathbf{r},t) = Ae^{-j(\omega t - \mathbf{k}\cdot\mathbf{r})}\,, \tag{1}$$

where ω is the temporal frequency of the plane wave.

The spatial derivative of the sound pressure field along the direction defined by vector $\mathbf{u}$ quantifies the rate of change of the sound pressure in that direction. It is given by the projection of the sound pressure field's spatial gradient onto the vector $\mathbf{u}$:

$$\begin{aligned} \mathrm{D}_{\mathbf{u}}(\mathbf{r},t) &= \nabla p(\mathbf{r},t)\cdot\mathbf{u} \\ &= j\,p(\mathbf{r},t)(\mathbf{k}\cdot\mathbf{u}) \\ &= jk\,\cos\theta\, p(\mathbf{r},t)\,, \end{aligned} \tag{2}$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{u}$.

Generalizing to the n-th-order spatial derivative along the direction defined by vector $\mathbf{u}$ gives:

$$\mathrm{D}^n_{\mathbf{u}}\, p(\mathbf{r},t) = (jk)^n\,(\cos\theta)^n\, p(\mathbf{r},t)\,. \tag{3}$$

The n-th-order spatial derivative of the sound pressure field composed of a plane wave has a bidirectional characteristic whose shape has the form

$$d^n(\theta) = A(\omega)(\cos\theta)^n\,, \tag{4}$$

where $A(\omega) = (jk)^n$ is a complex, frequency-dependent gain, and θ is the angle between the direction along which the spatial derivative is taken and the direction of propagation of a plane wave. The directional characteristics of a plane wave's spatial derivatives of different orders n along the positive direction x[1] are shown in Fig. 1.

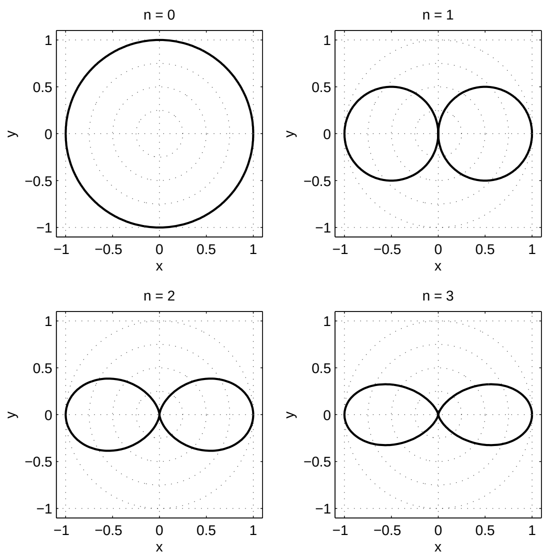

Fig. 1: Directional characteristic of plane wave spatial derivative, for different derivative orders n.

Also, the plane wave's n-th-order spatial derivative has at all angles a high-pass magnitude frequency characteristic proportional to $(jk)^n$ (or equivalently, $(j\omega/c)^n$), as shown in Fig. 2.

[1] The directional characteristics are plotted in the plane $z = 0$.

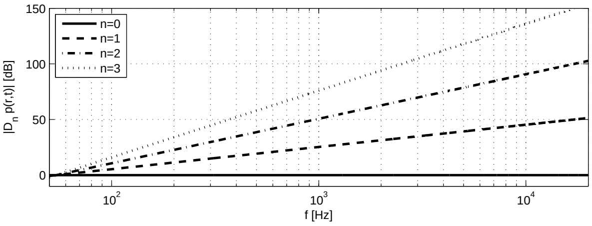

Fig. 2: Magnitude responses of a plane wave's spatial derivatives of different orders n.

2.2. Sound pressure field's spatio-temporal derivatives

Without loss of generality, the analysis will be given in a two-dimensional plane, such that the pressure field can be written as a function $p(x, y, t)$ of two spatial coordinates, x and y, and one temporal coordinate t. Since such pressure field is a function of three independent coordinates, its gradient is given by

$$\nabla p(x, y, t) = \left[\frac{\partial p}{\partial x} \; \frac{\partial p}{\partial y} \; \frac{\partial p}{\partial t} \right]^T . \quad (5)$$

The following analysis is given for a sound field composed of a plane wave with temporal frequency ω and wave vector $\mathbf{k} = (k \cos\theta \;\; k \sin\theta)^T$, given by the expression

$$p(x, y, t) = A\, e^{-j(\omega t - kx\cos\theta - ky\sin\theta)} , \quad (6)$$

where the angle θ defines the direction of propagation of a plane wave.

2.2.1. Sound pressure field's first-order spatio-temporal derivatives

The gradient of the sound pressure field given in (6) has the form

$$\nabla p(x, y, t) = jk\, p(x, y, t) \; [\cos\theta \quad \sin\theta \quad c]^T , \quad (7)$$

where c is the speed of sound propagation ($k = \omega/c$).

Let the unit vector $\mathbf{u}$, onto which the pressure gradient is projected, be defined as

$$\mathbf{u} = [\rho_u \cos\phi_u \quad \rho_u \sin\phi_u \quad u_t]^T , \quad (8)$$

where ρ_u ($\rho_u \in [0,\, 1]$) and ϕ_u ($\phi_u \in [0,\, 2\pi]$) define the spatial coordinates, and u_t ($u_t \in [0,\, 1]$) the temporal coordinate of the vector $\mathbf{u}$. Note that, since the vector $\mathbf{u}$ has a unit norm, $\rho_u^2 + u_t^2 = 1$, the ratio ρ_u/u_t gives the relation between its spatial part and its temporal part.

The sound pressure field's derivative along the spatio-temporal direction defined by the vector $\mathbf{u}$ is given by

$$\begin{aligned} \mathrm{D}_u\, p(x, y, t) &= \nabla p(x, y, t) \cdot \mathbf{u} \quad (9) \\ &= jk\, p(x, y, t)\, (\rho_u \cos(\theta - \phi_u) + c u_t) . \end{aligned}$$

The directional characteristic of a spatio-temporal derivative of a plane wave with wave vector $\mathbf{k}$ and temporal frequency ω is a combination of a first-order (bidirectional) directional characteristic of its spatial gradient, given by the term $\rho_n \cos(\theta - \phi_u)$, and a zero-order (omnidirectional) directional characteristic of a temporal differentiator, given by the term cu_t.

The relative contributions of the two derivatives—spatial and temporal—given by the ratio ρ_u/u_t, determine the shape of the directional characteristic of a spatio-temporal derivative of a plane wave. In the two extreme cases, when $\rho_u = 0$ and $\rho_u = 1$, the directional response is omnidirectional and bidirectional, respectively. When $\rho_u = cu_t$, the directional response has a well known cardioid polar pattern, and when the value of ρ_u is smaller or bigger than cu_t, the directional response is a variation of a sub-cardioid or a "tailed cardioid", respectively. Some well known first-order polar patterns, resulting from different ρ_u/u_t ratios, are given in Table 1 and shown in Fig. 3.

Response type	ρ_u/u_t
Cardioid	c
Sub-cardioid	$(0, c)$
Hyper-cardioid	$3c$
Super-cardioid	$\frac{3-\sqrt{3}}{\sqrt{3}-1}c$

Table 1: Some well known first-order polar patterns expressed through the ratio ρ_u/u_t of the spatio-temporal derivative.

2.2.2. Sound pressure field's higher-order spatio-temporal derivatives

The expression for a general n-th-order spatio-temporal derivative of the sound pressure field along a single direction given by the vector $\mathbf{u}$ is obtained

Response type	ρ_{u1}/u_{t1}	ρ_{u2}/u_{t2}	$\Delta\phi = \phi_{u1} - \phi_{u2}$
Cardioid	c	c	0
Hyper-cardioid	$(\sqrt{6}-1)c$	$(\sqrt{6}+1)c$	π
Super-cardioid	$\frac{4-\sqrt{7}+\sqrt{8-3\sqrt{7}}}{\sqrt{7}-2+\sqrt{8-3\sqrt{7}}}c$	$\frac{4-\sqrt{7}-\sqrt{8-3\sqrt{7}}}{\sqrt{7}-2-\sqrt{8-3\sqrt{7}}}c$	0

Table 2: Some well known second-order polar patterns expressed through the ratios ρ_u/u_t and angle differences $\Delta\phi$ of the spatio-temporal gradient.

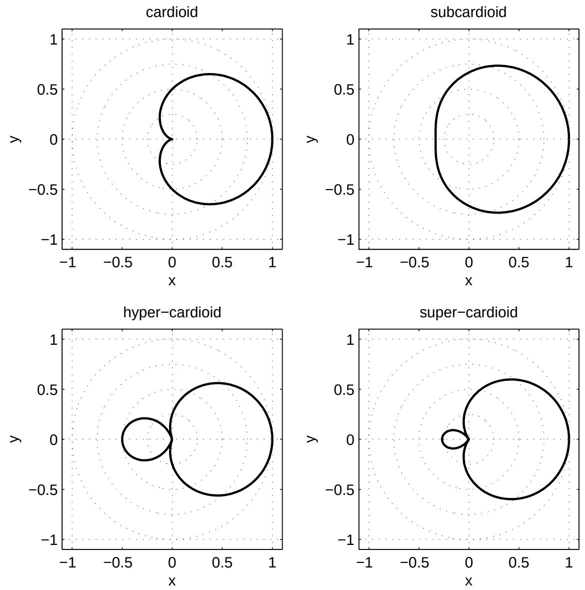

Fig. 3: Directional characteristics of a plane wave's first-order spatio-temporal derivatives for different ratios ρ_u/u_t, as given in Table 1.

by iterating the operations of taking its gradient and projecting it along the vector $\mathbf{u}$. For the sound field of a plane wave given by (6), the n-th-order spatio-temporal derivative along the direction given by (8) has the form

$$\mathrm{D}_u^n\, p(x,y,t) = (jk)^n \, \left(\rho_u \cos(\theta - \phi_u) + c u_t\right)^n \, p(x,y,t)\,. \tag{10}$$

Also, instead along a single direction, higher-order spatio-temporal derivatives can be taken along multiple directions. Given an n-tuple of vectors $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$, with each vector of the form $\mathbf{u}_i = [\rho_{ui}\cos\phi_{ui} \;\; \rho_{ui}\sin\phi_{ui} \;\; u_{ti}]^T$, a mixed derivative of the sound pressure field along directions given by $\mathbf{U}$ has the form

$$\mathrm{D}_U^n\, p(x,y,t) = (jk)^n \, p(x,y,t) \prod_{i=1}^{n} \left(\rho_{ui}\cos(\theta - \phi_{ui}) + c u_{ti}\right)\,. \tag{11}$$

As with the spatial derivative, the spatio-temporal derivative of the sound pressure field composed of a single plane wave has a high-pass frequency characteristic at all angles, which is proportional to $(jk)^n$, as shown in Fig. 2.

The directional characteristic is, however, proportional to a linear combination of spatial gradients of different orders, resulting from expanding the product $\prod_{i=1}^n (\rho_{ui}\cos(\theta - \phi_{ui}) + c u_{ti})$ in (11), or the term $(\rho_u \cos(\theta - \phi_u) + c u_t)^n$ in (10), which is a special case of (11).

As with the first-order, the shape of the directional characteristic of a higher-order spatio-temporal derivative of a plane wave sound field is determined by the choice of vectors $\mathbf{u}_i$, i.e., the parameters ρ_{ui}, ϕ_{ui} and u_{ti}. Some well known second-order polar patterns, resulting from different choices of ratios ρ_{u1}/u_{t1} and ρ_{u2}/u_{t2}, and angle differences $\Delta\phi = \phi_{u1} - \phi_{u2}$, are given in Table 2, and shown in Fig. 4.

3. PRACTICAL SPATIO-TEMPORAL DIFFERENTIAL MICROPHONE ARRAYS

Section 2 presented a theoretical analysis of a plane-wave sound field's spatio-temporal derivatives, which serves as a basis for designing gradient and differential microphone arrays with desired directional responses.

Practical differential microphone arrays are based on the principle of finite-difference approximation of a sound pressure field's spatio-temporal derivatives. They combine values of the sound pressure

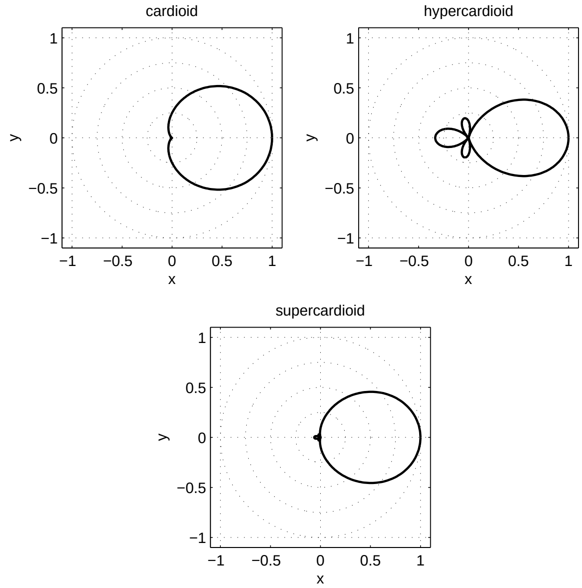

Fig. 4: Directional characteristics of a plane wave's second-order spatio-temporal derivatives for different ratios ρ_u/u_t and angle differences $\Delta\phi$, as given in Table 2.

field in multiple, closely-spaced points in space and time,[2] either acoustically (pressure at two faces of a diaphragm, and different-length acoustic paths to the two faces of a diaphragm) or electronically (pressure at different microphones of a microphone array combined with delay elements).

This section will present a few practical differential microphone array realizations based on the analysis from the previous section.

3.1. First-order differential microphone arrays: cardioid, hyper-cardioid, and super-cardioid

The first-order directional responses of sound field's spatio-temporal derivatives, presented in Section 2 and shown in Fig. 3, can be obtained by a finite-difference approximation of a sound field's spatio-temporal derivative, which involves taking differences of the sound pressure, both in space and in time. Two closely-spaced microphones, spaced at a distance d, together with a delay element, as shown in Fig. 5, can be used for a practical realization of any first-order differential microphone array.

[2]Points are spaced at a distance much shorter than the wavelength, and at a time much shorter than the period.

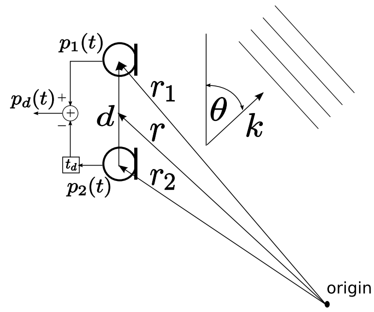

Fig. 5: First-order differential microphone realization using two pressure microphones and a delay element.

The response of a practical first-order differential microphone array, shown in Fig. 5, in a sound field of a plane wave, expressed in (1), is given by

$$p_d(t) = 2j \sin\left(\frac{k}{2}\left(d\cos\theta + ct_d\right)\right) p\left(\mathbf{r}, t - \frac{t_d}{2}\right), \quad (12)$$

where k is the wave number, d the inter-microphone distance, t_d the used delay, and $\mathbf{r}$ the position of the microphone array's center (mid-point between the two microphones). At low frequencies, (12) can be approximated by

$$p_d(t) \approx jk\left(d\cos\theta + ct_d\right)\; p\left(\mathbf{r}, t - \frac{t_d}{2}\right), \quad (13)$$

from which it can be seen that the ratio d/t_d determines the directional response of a practical differential microphone array in the same way the ratio ρ_u/u_t determines the directional response of the plane wave sound field's spatio-temporal derivative in (9).

Fig. 6 shows directional responses of the practical cardioid, super-cardioid and hyper-cardioid microphones realized with the microphone combination shown in Fig. 5, with $d = 2$ cm.

From Fig. 6, it can be seen that the shape of directional responses of practical first-order microphone arrays is frequency-dependent, and that it corresponds to the desired responses, shown in Fig. 3, only at low frequencies. Above the aliasing frequency[3], the directional characteristics deviate from

[3]The aliasing frequency of a first-order gradient microphone array is dependent on the inter-microphone distance d and the used delay t_d.

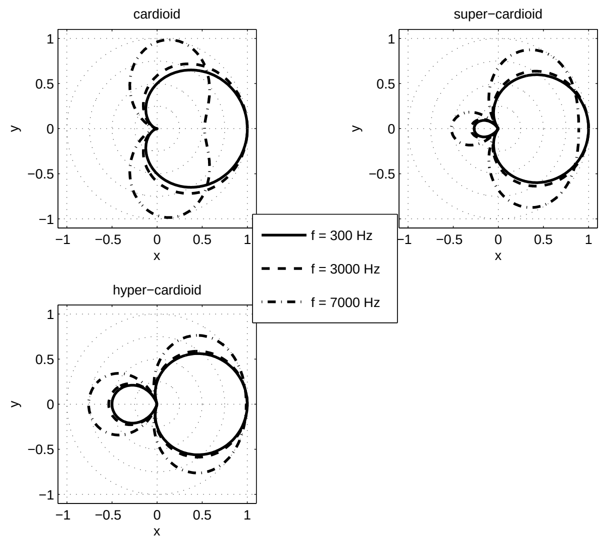

Fig. 6: Directional responses at various frequencies of first-order differential microphones realized as shown in Fig. 5, with $d = 2$ cm and $t_d = d/c$ (cardioid), $t_d = \frac{d(\sqrt{3}-1)}{c(3-\sqrt{3})}$ (super-cardioid) and $t_d = d/3c$ (hyper-cardioid).

the desired ones, as can be observed in Fig. 6 for frequency $f = 7000$ Hz.

3.2. Second-order differential microphone arrays

In this part, it is shown how clover-leaf directional responses $\sin 2\theta$ and $\cos 2\theta$ can be realized in two different ways based on the analysis from Section 2.

3.2.1. Clover-leaf response $\sin 2\theta$: quadrupole microphone array

The clover-leaf directional response $\sin 2\theta$ can be represented as a product of directional responses of two plane-wave sound field's spatial derivatives: the spatial derivative along the axis x, which has a directional response $\cos\theta$, and the spatial derivative along the axis y, which has a directional response $\sin\theta$ (or $\cos(\theta - \pi/2)$). As such, the directional response $\sin 2\theta$ can be realized as a cascade of two spatial derivative approximations: first along the axis x, and then along the axis y (or vice versa).

Fig. 7 illustrates a configuration of four pressure microphones used as an approximation of the previously described cascade of spatial derivatives of the sound field. Fig. 8 shows the directional responses at various frequencies of the quadrupole microphone array shown in Fig. 7, when the inter-microphone

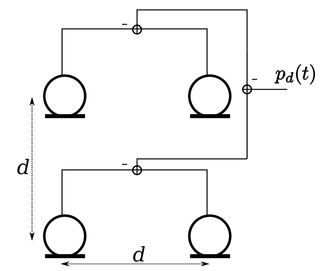

Fig. 7: Quadrupole microphone array used for obtaining a clover-leaf directional response of the form $\sin 2\theta$.

distance $d = 2$ cm is used.

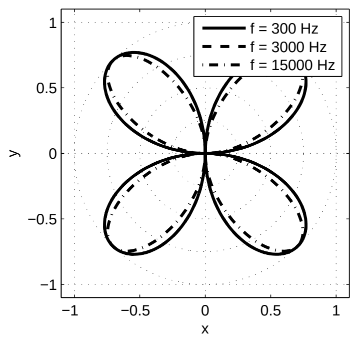

Fig. 8: Directional responses at various frequencies of the quadrupole microphone array shown in Fig. 7, with inter-microphone distance $d = 2$ cm.

3.2.2. Clover-leaf response $\cos 2\theta$: three-microphone line array

The clover-leaf directional response of the form $\cos 2\theta$ can be represented as

$$\cos 2\theta = 2\cos^2\theta - 1\,, \tag{14}$$

or equivalently, as

$$\cos 2\theta = (\sqrt{2}\cos\theta - 1)(\sqrt{2}\cos\theta + 1)\,, \tag{15}$$

which is a product of directional responses of two first-order spatio-temporal derivatives of a plane-wave sound pressure field. Consequently, the response $\cos 2\theta$ can be obtained by cascading two spatio-temporal derivative operations: one with $\rho_u/u_t = -\sqrt{2}$, and the other with $\rho_u/u_t = \sqrt{2}$, or equivalently, two spatio-temporal finite-differences: the first with $d/t_d = -\sqrt{2}c$ and the second with $d/t_d = \sqrt{2}c$ (or vice versa), as shown in Fig. 9.

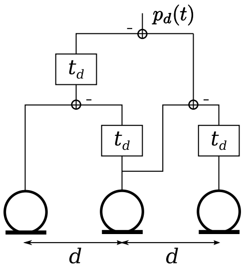

Fig. 9: A line-array with three microphones used to obtain the clover-leaf directional response $\cos 2\theta$.

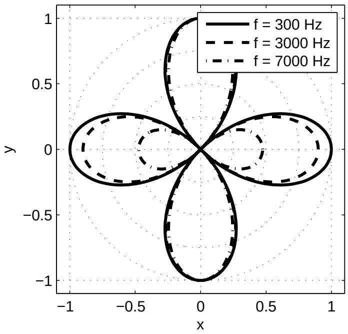

Fig. 10: Directional responses at various frequencies of the microphone array shown in Fig. 9, with inter-microphone distance $d = 2$ cm and inter-microphone delay $t_d = \frac{d}{\sqrt{2}c}$.

Fig. 10 shows the directional responses at various frequencies of the microphone array shown in Fig. 9, with the inter-microphone distance $d = 2$ cm and the delay $t_d = \frac{d}{\sqrt{2}c}$.

Like the first-order differential microphone arrays, the second-order differential microphone arrays have a directional response that is frequency-dependent. At low frequencies, it corresponds well to the desired response, and above the aliasing frequency, it deviates from the desired response. This can be observed in Fig. 10, which shows how the shape of the directional response of the microphone array from Fig. 9 deforms at the frequency $f = 7000$ Hz.

Note that the directional response of the form $\sin 2\theta$ can also be obtained by rotating by 45° the microphone array from Fig. 9.

4. CONCLUSIONS

This paper presented an analysis of the sound pressure field as a multivariate function of spatial location and time, which helps explaining the working principles of gradient microphones, differential microphones, and arrays as devices for approximately measuring sound pressure field's spatio-temporal derivatives, and shows their equivalence.

The presented analysis framework enables not only analyzing the response of a given gradient or differential microphone or microphone array, but it can also be used for designing differential microphone arrays. The appropriate adjustment of the microphone array's parameters—such as the array orientation and shape, the inter-microphone distances, and microphone signal delays—enables meeting the microphone array's desired response requirements.

5. REFERENCES

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